Closing Wed: HW_8 (8.3)

(Just one assignment, there is no 8A, 8B, 8C).

Midterm 2 will be returned Tuesday.

9.1 Introduction to Differential Equations

Goal: To see a few differential equations and understand what a solution is.

A **differential equation** is an equation involving derivatives.

Recall:

$\frac{dy}{dt}$ = "instantaneous rate of change of *y* with respect to *t* "

3 applied examples:

1. A simple model for population growth

Assume: "*The rate of growth of a population is proportional to the size of the population*."

P(t) = the population at year *t*,

$\frac{dP}{dt} =$ the rate of change of the population with respect to time (rate of growth).

So the assumption is equivalent to the differential equation

$$\frac{dP}{dt} = kP,$$

for some constant *k*.

2. Newton's Law of Cooling

Assume: "*The rate of cooling is proportional to the temperature difference between the object and its surroundings*."

T_s = constant temperature of the surroundings

T(t) = the temperature of an object at time *t*,

$\frac{dT}{dt}$ = the rate of change of the temperature with respect to time (rate of cooling).

$T - T_s$ = temp. difference between object and surroundings.

So Newton's Law of Cooling is equivalent to the differential equation

$$\frac{dT}{dt} = k(T - T_s),$$

for some constant *k*.

3. A Mixing Problem

Assume a 50 gallon vat is initially full of pure water. A salt water mixture is being dumped **into** the vat at 2 gal/min and this mixture contains 3 g/gal. The vat is mixed together.

At the same time, the mixture is coming **out** of the vat at 2 gal/min.

Let y(t) = grams of salt in the vat at time *t* min.

$\frac{y(t)}{50} =$ amount of salt per gal in the vat at time, t.

$\frac{dy}{dt} =$ the rate (g/min) at which amount of salt is changing with respect to time.

Salt is coming IN at a constant rate of

RATE IN = (3 g/gal)(2 gal/min) = 6 g/min

Salt is coming OUT at a rate of

RATE OUT = ($\frac{y}{50}$ g/gal)(2gal/min) = $\frac{y}{25}$ g/min

Thus,

$$\frac{dy}{dt} = 6 - \frac{y}{25}$$

A **solution to a differential equation** is any function that satisfies the equation.

Consider the differential equation: $\frac{dP}{dt} = 2P$

(a) $P(t) = 8e^{2t}$ is a solution because
$$\frac{dP}{dt} = 16e^{2t} \quad \text{and} \quad 2P = 16e^{2t},$$
which are the same.

(b) $P(t) = t^3$ is NOT a solution because
$$\frac{dP}{dt} = 3t^2 \quad \text{and} \quad 2P = 2t^3,$$
which are NOT the same.

(c) $P(t) = 0$ is a solution because
$$\frac{dP}{dt} = 0 \quad \text{and} \quad 2P = 0.$$

(d) The general solution is $P(t) = C\,e^{2t}$ (for any constant C). We will learn how to find this next time.

Consider the differential equation:

$y'' - 2y' + y = 0.$

(a) Is $y = e^{2t}$ a solution?

$y' = 2e^{2t}$ and $y'' = 4e^{2t}$

So $y'' - 2y' + y = 4e^{2t} - 4e^{2t} + e^{2t} = e^{2t}$,

which is NOT zero.

Thus it is NOT a solution.

(b) Is $y = t\,e^{t}$ a solution? YES (you check)

(c) There is a solution that looks like $y = e^{rt}$.

Can you find the value of *r* that works?

$y' = re^{rt}$, $y'' = r^2e^{rt}$ and we want

$y'' - 2y' + y = 0$ (for all values of t).

Substituting and factoring gives

$(r^2 - 2r + 1)e^{rt} = 0$, so we must have

$r^2 - 2r + 1 = 0$, which means $r = 1$.